

# Dirac field in topologically massive gravity

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26.May.2012 *file DiracMielkeBaekler06.tex*

## Abstract

We consider a Dirac field coupled minimally to the Mielke-Baekler model of gravity and investigate cosmological solutions in three dimensions. We arrive at a family of solutions which exists even in the limit of vanishing cosmological constant.

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# 1 Introduction

Gravity in three dimensions has attracted a lot of attention since the invention of the topologically massive gravity by Deser, Jackiw, Tempelton [1]-[3]. Another cornerstone is the discovery of the BTZ black hole solution to the Einstein's theory [4]. Since the inclusion of torsion makes the gravity models richer, Mielke and Baekler (MB) generalized the topologically massive gauge model of gravity by adding a new translational Chern-Simons term to the lagrangian of the standard topologically massive gravity [5]. There is a wide literature on the MB-model of gravity coupled to various material sources, see for example [6]-[10] and the references therein. We, however, realized that there is a tiny amount of work considering possible spinor couplings to the Einsteinian gravity and the standard topologically massive gravity in three dimensional spacetimes [11]-[14]. For the MB-case the situation is worse [15]. Thus we intend to fill in this gap to gain new insights to cosmological problems in the context of relationship between spinor and gravity.

We are interested in for investigating roles of a Dirac field in cosmology, because it is essential to a satisfactory description of relativistic fermions. Furthermore, a self-interacting or nonlinear Dirac field can yield negative pressure and thereby accelerate the early and the late-time expansion of the universe [16]. On the other hand, the authors show in the Ref.[17] that a single Dirac field can give rise to inflation within the four dimensional Einstein-Cartan theory, and prove compatibility of the Dirac-field model with the observations by calculating the power spectrum of density fluctuations of the Dirac field. Because of the inflationary (exponential) nature of our solution (33) this work may be seen in favor of those results.

The outline of the paper is as follows. Since we will be using the coordinate independent algebra of the exterior forms, in the Section 2 we fix our notations and conventions for the Riemann-Cartan spacetimes and a Dirac spinor in three dimensions. In the Section 3 after introducing the topological gauge lagrangian of gravity and the Dirac lagrangian we obtain the field equations by varying independently the total lagrangian with respect to the orthonormal basis 1-forms,  $e^a$ , the full connection 1-forms,  $\omega^{ab}$  and the adjoint of the Dirac field,  $\bar{\Psi}$ . In the Section 4 firstly we write down the homogenous and the isotropic metric in the plane polar coordinates and make torsion ansatz. Secondly we compute the Dirac equation and its adjoint. This allows us to calculate the Dirac energy-momentum 2-forms explicitly. Then we obtain all

field equations via the computer algebra system, the Reduce and the package Excalc [18],[19]. Thus we determine a class of solution for the homogenous and isotropic geometry and a certain Dirac spinor. We summarize and discuss the results in the Section 5.

## 2 Notations and Conventions

We specify the Riemann-Cartan space-time by a triplet  $(M, g, \nabla)$  where  $M$  is a 3-dimensional differentiable manifold equipped with a metric tensor

$$g = \eta_{ab} e^a \otimes e^b \quad (1)$$

of signature  $(-, +, +)$ .  $e^a$  is an orthonormal co-frame dual to the frame vectors  $X_a$ , that is  $e^a(X_b) \equiv \iota_b e^a = \delta_b^a$  where  $\iota_b := \iota_{X_b}$  denotes the interior product. A metric compatible connection  $\nabla$  can be specified in terms of connection 1-forms  $\omega^a_b$  satisfying  $\omega_{ba} = -\omega_{ab}$ . Then the Cartan structure equations

$$de^a + \omega^a_b \wedge e^b = T^a, \quad (2)$$

$$d\omega^a_b + \omega^a_c \wedge \omega^c_b = R^a_b \quad (3)$$

define the space-time torsion 2-forms  $T^a$  and curvature 2-forms  $R^a_b$ , respectively. Here  $d$  denotes the exterior derivative and  $\wedge$  the wedge product. These tensor-valued 2-forms satisfy the Bianchi identities

$$DT^a = R^a_b \wedge e^b, \quad DR^a_b = 0. \quad (4)$$

Because of the first Bianchi identity, the Ricci tensor does not need to be symmetric in the Riemann-Cartan spacetimes. We fix the orientation of space-time by choosing the volume 3-form  $*1 = e^0 \wedge e^1 \wedge e^2$  where  $*$  is the Hodge star map. In three dimensional space-times with Lorentz signature for any  $p$ -form  $** = -1$ . We will use the abbreviations  $e^{ab\dots} := e^a \wedge e^b \wedge \dots$  and  $\iota_{ab\dots} := \iota_a \iota_b \dots$ . It is possible to decompose the connection 1-forms in a unique way as

$$\omega^a_b = \hat{\omega}^a_b + K^a_b \quad (5)$$

where  $\hat{\omega}^a_b$  are the zero-torsion Levi-Civita connection 1-forms satisfying

$$de^a + \hat{\omega}^a_b \wedge e^b = 0 \quad (6)$$

and  $K^a{}_b$  are the contortion 1-forms satisfying

$$K^a{}_b \wedge e^b = T^a. \quad (7)$$

The curvature 2-forms are also decomposed in a similar way:

$$R^a{}_b = \hat{R}^a{}_b + \hat{D}K^a{}_b + K^a{}_c \wedge K^c{}_b \quad (8)$$

with

$$\hat{D}K^a{}_b = dK^a{}_b + \hat{\omega}^a{}_c \wedge K^c{}_b - \hat{\omega}^c{}_b \wedge K^a{}_c.$$

We are using the formalism of Clifford algebra  $\mathcal{Cl}_{1,2}$ -valued exterior forms.  $\mathcal{Cl}_{1,2}$  algebra is generated by the relation among the orthonormal basis  $\{\gamma_0, \gamma_1, \gamma_2\}$

$$\gamma^a \gamma^b + \gamma^b \gamma^a = 2\eta^{ab}. \quad (9)$$

One particular representation of the  $\gamma^a$ 's is given by the following Dirac matrices

$$\gamma_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \gamma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (10)$$

In this case a Dirac spinor  $\Psi$  can be represented by a 2-component column matrix. Thus we write explicitly the covariant exterior derivative of  $\Psi$ , its Dirac conjugate and the curvature of the spinor bundle, respectively,

$$D\Psi = d\Psi + \frac{1}{2}\sigma_{ab}\Psi\omega^{ab}, \quad D\bar{\Psi} = d\bar{\Psi} - \frac{1}{2}\bar{\Psi}\sigma_{ab}\omega^{ab}, \quad D^2\Psi = \frac{1}{2}R^{ab}\sigma_{ab}\Psi \quad (11)$$

where  $\sigma_{ab} := \frac{1}{4}[\gamma_a, \gamma_b] = \frac{1}{2}\epsilon_{abc}\gamma^c$  are the generators of the Lorentz group  $SO(1,2)$ . The Dirac adjoint is  $\bar{\Psi} := \Psi^\dagger\gamma_0$ . We frequently make use of the identity

$$\gamma_c\sigma_{ab} + \sigma_{ab}\gamma_c = \epsilon_{abc}. \quad (12)$$

### 3 Field Equations

The field equations of our model are obtained by varying the action

$$I[e^a, \omega^{ab}, \bar{\Psi}] = \int_M (L_G + L_D) \quad (13)$$

where the gravitational lagrangian density 3-form is given by Mielke and Baekler [5]

$$L_G = \frac{a}{2} R_{ab} \wedge {}^* e^{ab} + \lambda {}^* 1 + \frac{b}{2} T^a \wedge e_a + \frac{c}{2} (\omega^a{}_b \wedge d\omega^b{}_a + \frac{2}{3} \omega^a{}_b \wedge \omega^b{}_c \wedge \omega^c{}_a) \quad (14)$$

and the (hermitian) Dirac lagrangian density 3-form

$$L_D = \frac{i}{2} (\bar{\Psi} {}^* \gamma \wedge D\Psi - D\bar{\Psi} \wedge {}^* \gamma \Psi) + im \bar{\Psi} \Psi {}^* 1. \quad (15)$$

Here the gravitational constants  $a$ , mass  $m$  and the Dirac field  $\Psi$  have the dimension of  $length^{-1}$ , the gravitational constant  $b$  has the dimension of  $length^{-2}$ ,  $c$  is dimensionless constant, and the cosmological constant  $\lambda$  has the dimension of  $length^{-3}$ . The case  $b = 0$  corresponds the topologically massive gravity with cosmological constant. The hermiticity of the lagrangian (15) leads to a charge current which admits the usual probabilistic interpretation.  $\frac{b}{2} T^a \wedge e_a$  is known as the *torsional Chern-Simons term* which corresponds the usual Chern-Simons 3-form for the curvature,  $(1/2)(\omega^a{}_b \wedge d\omega^b{}_a + (2/3)\omega^a{}_b \wedge \omega^b{}_c \wedge \omega^c{}_a)$ , for the curvature.

We obtain the field equations via independent variations with respect to  $e^a, \omega^{ab}, \bar{\Psi}$ . Thus  $e^a$ -variation yields the FIRST equation

$$-\frac{a}{2} \epsilon_{abc} R^{bc} - \lambda {}^* e_a - b T_a = \tau_a, \quad (16)$$

$\omega^{ab}$ -variation yields the SECOND equation

$$-\frac{a}{2} \epsilon_{abc} T^c + \frac{b}{2} e_{ab} + c R_{ab} = \Sigma_{ab}, \quad (17)$$

and  $\bar{\Psi}$ -variation yields the Dirac equation

$${}^* \gamma \wedge (D - \frac{1}{2} \mathcal{V}) \Psi + m \Psi {}^* 1 = 0, \quad (18)$$

where  $\Sigma_{ab} = -\frac{i}{4} \bar{\Psi} \Psi e_{ab}$  is the Dirac spin angular momentum 2-form and  $\tau_a$  is the Dirac energy-momentum 2-form

$$\tau_a = \frac{i}{2} {}^* e_{ba} \wedge [\bar{\Psi} \gamma^b (D\Psi) - (D\bar{\Psi}) \gamma^b \Psi] + im \bar{\Psi} \Psi {}^* e_a. \quad (19)$$

For future convenience by using the Dirac equation (18) and its conjugate  $(D - \frac{1}{2}\mathcal{V})\bar{\Psi} \wedge *\gamma - m\bar{\Psi}*1 = 0$  we rewrite the Dirac energy-momentum and the spin angular momentum 2-forms as

$$\tau_a = -\frac{i}{2} [\bar{\Psi}\gamma_b(\partial_a\Psi) - (\partial_a\bar{\Psi})\gamma_b\Psi] *e^b + \mathcal{S}\omega_{bc,a}e^{bc} \quad (20)$$

$$\Sigma_{ab} = -\mathcal{S}e_{ab} \quad (21)$$

where  $\partial_a := \iota_a d$ ,  $\omega_{bc,a} := \iota_a \omega_{bc}$  and  $\mathcal{S} := \frac{i}{4}\bar{\Psi}\Psi$ .

## 4 A Class of Cosmological Solution

We consider the homogeneous and isotropic metric in the plane polar coordinates  $(t, r, \phi)$

$$g = -dt^2 + \frac{Q^2(t)}{1 - kr^2}dr^2 + r^2Q^2(t)d\phi^2 \quad (22)$$

with the expansion factor  $Q(t)$  and curvature index  $k$ . We choose the orthonormal basis 1-forms

$$e^0 = dt, \quad e^1 = \frac{Q(t)}{\sqrt{1 - kr^2}}dr, \quad e^2 = rQ(t)d\phi, \quad (23)$$

leading to the Levi-Civita connection 1-forms

$$\hat{\omega}^0_1 = \frac{\dot{Q}}{Q}e^1, \quad \hat{\omega}^0_2 = \frac{\dot{Q}}{Q}e^2, \quad \hat{\omega}^1_2 = -\frac{\sqrt{1 - kr^2}}{rQ}e^2 \quad (24)$$

where dot denotes the derivative with respect to  $t$ . Two independent functions are enough to describe the most general torsion preserving homogeneity and isotropy of the spacetime [20]. Thus we choose the torsion

$$T^0 = u(t)e^{01} + v(t)e^{12}, \quad T^1 = v(t)e^{02}, \quad T^2 = -v(t)e^{01} - u(t)e^{12}. \quad (25)$$

We then calculate the contortion 1-forms via (7)

$$K^0_1 = ue^0 - \frac{v}{2}e^2, \quad K^0_2 = \frac{v}{2}e^1, \quad K^1_2 = -ue^2 + \frac{v}{2}e^0. \quad (26)$$

Now we write explicitly the full connection 1-forms with the help of the equation (5)

$$\begin{aligned}\omega_{01} &= -ue^0 - \frac{\dot{Q}}{Q}e^1 + \frac{v}{2}e^2, & \omega_{02} &= -\frac{v}{2}e^1 - \frac{\dot{Q}}{Q}e^2, \\ \omega_{12} &= \frac{v}{2}e^0 - \left(u + \frac{\sqrt{1-kr^2}}{rQ}\right)e^2.\end{aligned}\quad (27)$$

Thus the curvature 2-forms are computed as

$$\begin{aligned}R^0{}_1 &= \frac{(4\ddot{Q} + v^2Q)e^{01} - 2\dot{v}Qe^{02} + 2uvQe^{12}}{4Q}, \\ R^0{}_2 &= \frac{2r\dot{v}Qe^{01} - \left[4u\sqrt{1-kr^2} + r(4u^2 - v^2)Q - 4r\ddot{Q}\right]e^{02} - 4ru\dot{Q}e^{12}}{4rQ}, \\ R^1{}_2 &= \frac{2ruvQ^2e^{01} - 4r\dot{u}Q^2e^{02} + \left[4r\dot{Q}^2 + rv^2Q^2 + 4kr - 4uQ\sqrt{1-kr^2}\right]e^{12}}{4rQ^2}.\end{aligned}\quad (28)$$

For the consistency of the equations, that is, for keeping all  $Q, u, v$  functions to be dependent only on the cosmic time  $t$  we have to assume  $\Psi$  depending on both  $t$  and  $r$  such that  $\Psi = \Psi(t, r) = \xi(t)/\sqrt{r}$ . Thus the Dirac equation (18) and its conjugate turn out to be

$$\dot{\xi} = \left[-\dot{Q}/Q + (3v/4 - m)\gamma_0\right]\xi, \quad (29)$$

$$\dot{\bar{\xi}} = \bar{\xi} \left[-\dot{Q}/Q - (3v/4 - m)\gamma_0\right]. \quad (30)$$

These results enable us to determine the Dirac energy-momentum 2-forms (20)

$$\begin{aligned}\tau_0 &= 2\mathcal{S}[-ue^{01} + (2m - v)e^{12}], \\ \tau_1 &= -\frac{\mathcal{S}}{Q}[2\dot{Q}e^{01} + vQe^{02}], \\ \tau_2 &= \frac{\mathcal{S}}{rQ}[rvQe^{01} - 2r\dot{Q}e^{02} - 2(ruQ + \sqrt{1-kr^2})e^{12}].\end{aligned}\quad (31)$$

Consequently we obtain the following class of solution to the field equations (16) and (17)

$$u = 0, \quad v = \frac{ab - 2c\lambda}{a^2 - 2bc}, \quad \mathcal{S} = 0, \quad (32)$$

$$Q(t) = \frac{1}{2h} [e^{h(t+c_1)} + ke^{-h(t+c_1)}] \quad (33)$$

where  $c_1$  is a constant and  $h^2 = [4(a^2 - 2bc)(b^2 - a\lambda) - (ab - 2c\lambda)^2]/4(a^2 - 2bc)^2$ . If  $k < 0$  and  $c_1 = (\ln |k|)/(2h)$ , the space-time characterized by (33) which recalls the inflationary solution of the General Relativity in four dimensions has a singularity, i.e.  $Q(0) = 0$ . Here we also notice that it must be  $a^2 - 2bc \neq 0$  for the solution, but even if  $\lambda = 0$  we have a solution. The condition  $a^2 - 2bc \neq 0$  was shown to be arising from the demands of a canonical constrained hamiltonian analysis in [21]. For more readings one can consult the equation (38) and the succeeding remarks of that work with the replacement  $a \rightarrow a$ ,  $\alpha_3 \rightarrow 2c$  and  $\alpha_4 \rightarrow b$ . However, a prescription to approach the singular point  $a^2 = 2bc$  and the canonical structure analysis of the theory in the space of parameters is discussed in [22].

The next task is to solve the Dirac equation (29)

$$\begin{pmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \end{pmatrix} = \begin{pmatrix} -\dot{Q}/Q & 3v/4 - m \\ -3v/4 + m & -\dot{Q}/Q \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}. \quad (34)$$

We can decouple these equations by virtue of a unitary transformation

$$\xi = U\rho \quad \text{with} \quad U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}. \quad (35)$$

Thus we arrive at the set of decoupled equations

$$\dot{\rho}_1 = [-\dot{Q}/Q + i(3v/4 - m)] \rho_1, \quad \dot{\rho}_2 = [-\dot{Q}/Q - i(3v/4 - m)] \rho_2. \quad (36)$$

The solutions can be written as

$$\rho_1(t) = \frac{\rho_+}{Q(t)} e^{+i\phi(t)}, \quad \rho_2(t) = \frac{\rho_-}{Q(t)} e^{-i\phi(t)} \quad (37)$$

where  $\rho_{\pm}$  are the integration constants and  $\phi = (3v/4 - m)t$  is the phase factor. Then we calculate the Dirac condensate that is zero according to (32iii)

$$\mathcal{S} = \frac{i}{4} \bar{\Psi} \Psi = \frac{i}{4r} \bar{\xi} \xi = \frac{|\rho_-|^2 - |\rho_+|^2}{4rQ^2} = 0. \quad (38)$$

Consequently it must be  $|\rho_+|^2 = |\rho_-|^2$ . Furthermore since  $\mathcal{S} = 0$ , the Dirac spin 3-form is zero,  $\Sigma_{ab} = 0$ , via (21) as well.



## 5 Concluding Remarks

In this work we considered the minimal coupling of a Dirac particle to the MB-model of gravity in three dimensional Riemann-Cartan spacetime. After computing the variational field equations we considered the homogenous and isotropic metric, and made an ansatz for torsion. Then we calculated the Dirac equation which gave us the opportunity of obtaining the Dirac energy-momentum 2-forms. Consequently we were at a position of writing all field equations explicitly. Thus we obtained a family of solution (32), (33) and (37) which is valid for  $a^2 - 2bc \neq 0$  and even if  $\lambda = 0$ .

It is worthwhile to pay more attention to the result,  $\mathcal{S} = 0$ , given by the equation (32). It may imply that even though the Dirac field has no effect on the geometry, the later affects the time evolution of the spinor field. Thus our solution may evoke the Einsteinian cosmological solution satisfying perfect cosmological principle in which  $\lambda \neq 0$  is a crucial condition. However, our model and the geometry are totally different from Einstein's theory and the Riemann spacetime, respectively. In fact, since the presence of a Dirac spinor modifies the geometry of space-time by affecting the connection that determines the notion of parallel transport, it is no longer possible to conclude that the geometry is totally unaware of the presence of a Dirac spinor.

Meanwhile, there are works in which the Einstein-Cartan-Dirac (ECD) theory was investigated in four dimensions, for example [23],[24],[25]. Even they were done in four dimensions, they are related to our work, especially, with regards to  $\mathcal{S} = 0$ . For clarifying the relation we firstly notice that  $\mathcal{S} = 0$  means  $*(\Sigma_{ab} \wedge *\Sigma^{ab}) = 0$  equivalent to  $\Sigma_{abc}\Sigma^{abc} = 0$  in the language of tensors with the notation  $\Sigma_{ab} := \Sigma_{abc} *e^c$ . In four dimensions because of the complexity of the ECD problem the authors of those papers proceed with the following strategy. They firstly determine orthonormal coframe  $e^a$  by solving the Einstein equation (with cosmological constant) in vacua. Then they give the constraint by hand  $\Sigma_{abc}\Sigma^{abc} = 0$  for  $\Psi$ . Finally they compute torsion and spinor through the field equations. For massless Dirac spinors this constraint corresponds to the helicity state equation  $\frac{1}{2}(1 \pm i\gamma_5)\Psi = \Psi$  [23]. The neutrinos are considered as massless fermions and correspondingly have definite helicities according to the standard model. Therefore, they may be treated in this context. Thus in the Ref.[26] even though the authors did not follow directly the strategy outlined above, their results and calculations are parallel to it. However, in three dimensions because of relative simplicity

one does not need to assume any constraint like that from the beginning. We encounter with that as a result of the equations. But this is not a generic result, that is, for some other models and their solutions in (1+2)-dimensions it may be  $\Sigma_{abc}\Sigma^{abc} \approx \mathcal{S}^2 \neq 0$ , see e.g. [14].

## Appendix: Irreducible Decompositions

In this section we give briefly the irreducible pieces of torsion, contortion and curvature in three dimensions. Firstly torsion which has nine components can be decomposed

$$\underbrace{T^a}_{\#9} = \underbrace{{}^{(1)}T^a}_{\#5} + \underbrace{{}^{(2)}T^a}_{\#3} + \underbrace{{}^{(3)}T^a}_{\#1} \quad (39)$$

where  ${}^{(2)}T^a = -\frac{1}{2}(\iota_b T^b) \wedge e^a$ ,  ${}^{(3)}T^a = \frac{1}{3}\iota^a(e_b \wedge T^b)$  and  ${}^{(1)}T^a = T^a - {}^{(2)}T^a - {}^{(3)}T^a$ . In this section the notation with the number under a brace is for the number of components of that part. They have the properties,  ${}^{(1)}T^a \wedge e_a = {}^{(2)}T^a \wedge e_a = 0$  and  $\iota_a {}^{(1)}T^a = \iota_a {}^{(3)}T^a = 0$ . Our choice (25) corresponds to

$${}^{(1)}T^a = 0, \quad {}^{(2)}T^a = u \begin{pmatrix} e^{01} \\ 0 \\ -e^{12} \end{pmatrix}, \quad {}^{(3)}T^a = v \begin{pmatrix} e^{12} \\ e^{02} \\ -e^{01} \end{pmatrix} \quad (40)$$

which means  $4 = 3 \oplus 1$ . After the solution (32) we are left only with  ${}^{(3)}T^a$ .

Secondly one can decompose the contortion having nine components

$$\underbrace{K_{ab}}_{\#9} = \underbrace{{}^{(1)}K_{ab}}_{\#5} + \underbrace{{}^{(2)}K_{ab}}_{\#3} + \underbrace{{}^{(3)}K_{ab}}_{\#1} \quad (41)$$

where  ${}^{(2)}K_{ab} = \frac{1}{2}[e_a \wedge (\iota^c K_{cb}) - e_b \wedge (\iota^c K_{ca})]$ ,  ${}^{(3)}K_{ab} = -\frac{1}{6}\iota_{ab}(K_{cd} \wedge e^{cd})$  and  ${}^{(1)}K_{ab} = K_{ab} - {}^{(2)}K_{ab} - {}^{(3)}K_{ab}$ . They have the properties  $\iota_a {}^{(1)}K^{ab} = \iota_a {}^{(3)}K^{ab} = 0$  and  ${}^{(1)}K_{ab} \wedge e^{ab} = {}^{(2)}K_{ab} \wedge e^{ab} = 0$ . For our case (26) we possess again  ${}^{(1)}K_{ab} = 0$ , but nonzero  ${}^{(2)}K_{ab}$  and  ${}^{(3)}K_{ab}$ , i.e.  $4 = 3 \oplus 1$ . Besides after the solution (32) only  ${}^{(3)}K_{ab}$  survives.

Finally one can split the curvature with nine components

$$\underbrace{R_{ab}}_{\#9} = \underbrace{{}^{(1)}R_{ab}}_{\#5} + \underbrace{{}^{(2)}R_{ab}}_{\#3} + \underbrace{{}^{(3)}R_{ab}}_{\#1} \quad (42)$$

where  ${}^{(2)}R_{ab} = \frac{1}{2}(e_a \wedge \iota_b - e_b \wedge \iota_a)(e^c \wedge R_c)$ ,  ${}^{(3)}R_{ab} = \frac{1}{6}Re_{ab}$  and  ${}^{(1)}R_{ab} = R_{ab} - {}^{(2)}R_{ab} - {}^{(3)}R_{ab}$  with  $R_a = \iota^b R_{ba}$  and  $R = \iota^a R_a$ . They have the properties  $\iota_{ab}{}^{(1)}R^{ab} = \iota_{ab}{}^{(2)}R^{ab} = 0$ ,  ${}^{(1)}R_{ab} \wedge e^b = {}^{(3)}R_{ab} \wedge e^b = 0$  and  $e_b \wedge \iota_a{}^{(1)}R^{ab} = 0$ . For the solution (32) although all three pieces of curvature and of torsion are nonzero, since  $DT^a = 0$  the Ricci tensor is symmetric.

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